NON-RANDOM OVERSHOOTS OF LÉVY PROCESSES

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Abstract. The class of Lévy processes for which overshoots are almost surely constant quantities is precisely characterized.

1. Introduction

Fluctuation theory represents one of the most important areas within the study of Lévy processes, with applications in finance, insurance, dam theory etc. [9] A key result, then, is the Wiener-Hopf factorization, particularly explicit in the spectrally negative case, when there are no positive jumps, a.s. [12, Section 9.46] [2, Chapter VII].

What makes the analysis so much easier in the latter instance, is the fact that the overshoots [12, p. 369] over a given level are known *a priori* to be constant and equal to zero. As we shall see, this is also the only class of Lévy process for which this is true (see Lemma 6). But it is not so much the exact values of the overshoots that matter, as does the fact that these values are non-random (and known). It is therefore natural to ask if there are any other Lévy processes having constant overshoots (a.s.) and, moreover, what *precisely* is the class having this property.

To be sure, in the existing literature one finds expressions regarding the distribution of the overshoots. For example, in [12, p. 369, Theorem 49.1] one has given the joint Laplace transform of this quantity. Similarly, in [3] we find an expression for the law of the overshoot in terms of the Lévy measure, but only after it has been integrated against the bivariate renewal functions. Unfortunately, both of these do not seem immediately useful in answering the question posed above.

Further to this, the asymptotic study of quantities at first passage above a given level has been undertaken in [3, 10] and behaviour just prior to first passage has also been investigated, see, e.g. [12, p. 378, Remark 49.9] and [9, Chapter 7]. On the other hand it appears that the (grantedly not very difficult) question, as above, has not yet received due attention.

The answer to it, presented in this paper, is as follows: for the overshoots of a Lévy process to be (conditionally on the process going above the level in question) almost surely constant quantities, it is both necessary and sufficient that *either* the process has no positive jumps (a.s.) *or* for some $h > 0$, it is compound Poisson, living on the lattice $\mathbb{Z}_h := h\mathbb{Z}$, and which can only jump up by h.

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The precise and more exhaustive statement of this result is contained in Theorem 3 of Section 2, which also introduces the required notation. Section 3 supplies the proof and Section 4 concludes. Finally, the appendix contains a result concerning conditional expectation, which is used in the proof, but also interesting in its own right.

2. Notation and statement of result

Throughout we work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, which satisfies the standard assumptions (i.e. the σ-field $\mathcal{F}$ is P-complete and the filtration $\mathbb{F}$ is right continuous and $\mathcal{F}_0$ contains all P-null sets). We let X be a Lévy process on this space with characteristic triplet (σ^2, λ, μ) relative to some cut-off function [12]. $\overline{X}_t := \sup\{X_s : s \in [0,t]\}$ $(t \geq 0)$ is the supremum process of X.

Next, for $x \in \mathbb{R}$ introduce $T_x := \inf\{t \geq 0 : X_t \geq x\}$ (resp. $\hat{T}_x := \inf\{t \geq 0 : X_t > x\}$), the *first entrance time* of X to $[x, \infty)$ (resp. (x, ∞)). We will informally refer to T_x and $\hat{T}_x$ as the *times of first passage* above the level x.

$\mathcal{B}(S)$ will always denote the Borel σ-field of a topological space S, $\mathrm{supp}(m)$ the support of a measure m thereon. For a random element $R : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$, $R_\star\mathsf{P}$ is the image measure. $\overline{\mathcal{S}}^\mu$ denotes the completion of the σ-field $\mathcal{S}$ relative to the measure μ and $\mathcal{S}^\star$ is the universal completion.

Definition 1 (Upwards-skip-free Lévy chain)**.** *A Lévy process X is an* upwards-skip-free Lévy chain *if it is a compound Poisson process, and for some (then unique) $h > 0$, $\mathrm{supp}(\lambda) \subset \mathbb{Z}_h$ and $\mathrm{supp}(\lambda|_{\mathcal{B}((0,\infty))}) = \{h\}$.*

Finally, the following notion, which is a rephrasing of "being almost surely constant conditionally on a given event", will prove useful:

Definition 2 (P-triviality)**.** *Let $S \neq \emptyset$ be any measurable space, whose σ-algebra contains the singletons. An S-valued random element R is said to be* P-trivial *on an event $A \in \mathcal{F}$ if there exists $r \in S$ such that $R = r$ a.s.-P on A (i.e. the push-forward measure $R|_{A*}\mathsf{P}(\cdot \cap A)$ is a weighted (possibly by 0, if $\mathsf{P}(A) = 0$) δ-measure). R may only be defined on some $B \supset A$.*

We can now state succinctly the main result of this paper:

Theorem 3 (Non-random position at first passage time)**.** *The following are equivalent:*

(a) For some $x > 0$, $X(T_x)$ is P-trivial on $\{T_x < \infty\}$.
(b) For all $x \in \mathbb{R}$, $X(T_x)$ is P-trivial on $\{T_x < \infty\}$.
(c) For some $x \geq 0$, $X(\hat{T}_x)$ is P-trivial on $\{\hat{T}_x < \infty\}$ and a.s.-P positive thereon (in particular the latter obtains if $x > 0$).
(d) For all $x \in \mathbb{R}$, $X(\hat{T}_x)$ is P-trivial on $\{\hat{T}_x < \infty\}$.
(e) Either X has no positive jumps, a.s.-P or X is an upwards-skip-free Lévy chain.

If so, then outside a P-negligible set, for each $x \in \mathbb{R}$, $X(T_x)$ (resp. $X(\hat{T}_x)$) is constant on $\{T_x < \infty\}$ (resp. $\{\hat{T}_x < \infty\}$), i.e. the exceptional set in (b) (resp. (d)*) can be chosen not to depend on x.*

Finally, notation-wise, we make the following explicit: $\mathbb{R}^+ := (0,\infty)$, $\mathbb{R}_+ := [0,\infty)$, $\mathbb{R}^- := (-\infty,0)$ and $\mathbb{R}_- := (-\infty,0]$; for $q \in (0,\infty)$, $\mathrm{Exp}(q)$ denotes the exponential law (mean $1/q$); the symbol $\perp$ is sometimes used to indicate stochastic independence (relative to the probability measure P).

3. Proof of theorem

Remark 4. *T_x and $\hat{T}_x$ are $\mathbb{F}$-stopping times for each $x \in \mathbb{R}$ (apply the début theorem* [6, p. 101, Theorem 6.7]*) and $\mathsf{P}(\forall x \in \mathbb{R}_-(T_x = 0)) = 1$. Moreover, $\mathsf{P}(\forall x \in \mathbb{R}(T_x < \infty)) = 1$, whenever X either drifts to $+\infty$ or oscillates. If not, then it drifts to $-\infty$* [12, p. 255, Proposition 37.10] *and on the event $\{T_x = \infty\}$ one has $\lim_{t\to T_x} X(t) = -\infty$ for all $x \in \mathbb{R}$, a.s.-P.*

For the most part we find it more convenient to deal with the $(T_x)_{x\in\mathbb{R}}$, rather than $(\hat{T}_x)_{x\in\mathbb{R}}$, even though this makes certain measurability issues more involved.

Remark 5. *Note that whenever 0 is regular for $(0,\infty)$, then for each $x \in \mathbb{R}$, $T_x = \hat{T}_x$ a.s.-P (apply the the strong Markov property* [12, p. 278, Theorem 40.10] *at the time T_x). For conditions equivalent to this, see* [9, p. 142, Theorem 6.5]. *Conversely, if 0 is irregular for $(0,\infty)$, then with a positive P-probability $\hat{T}_0 > 0 = T_0$.*

Lemma 6 (Continuity of the running supremum)**.** *The supremum process $\overline{X}$ is continuous (P-a.s.) iff X has no positive jumps (P-a.s). In particular, if $X(T_x) = x$ a.s.-P on $\{T_x < \infty\}$ for each $x > 0$, then X has no positive jumps, a.s.-P.*[1]

Proof. We first show the validity of the equivalence. Sufficiency of the "no positive jumps" condition is immediate. For necessity, prove by contradiction: suppose *per absurdum* X had positive jumps with a positive probability and its supremum process would continue to enjoy the property of continuity, a.s.-P. Then for some $a > 0$, X would have a jump exceeding a with a positive probability. By the Lévy-Itô decomposition, one may write, P-a.s., $X = X^1 + X^2 + X^3$ as an independent sum of a Brownian motion with drift X^1, a compound Poisson process X^2 of the positive jumps of X exceeding (i.e. of height $>$) a and finally whatever remains X^3 (see e.g. [1, p. 108, Theorem 2.4.16] and the results leading thereto, in particular [1, p. 99, Theorem 2.4.6]). Let Z be the supremum process of $|X^1 + X^3|$. Note that by right continuity of the sample paths, for some $t > 0$, $\mathsf{P}(\{Z_t < a/2\}) > 0$. Let T be the first jump time of X^2. Then, by independence, and the fact that $T \sim \mathrm{Exp}(\lambda(a,\infty))$ [1, p. 101, Theorem 2.3.5(1)], one has $\mathsf{P}(\{Z_t < a/2\} \cap \{T < t\}) > 0$.

[1] Measurability of these events is a consequence of the following (it is sufficient to have X $\mathbb{F}$-adapted and càdlàg (P-a.s., if the probability space is complete)). For each $\epsilon > 0$, let $A_\epsilon := \cup_{s\in[0,t]}\{X_s - X_{s-} > \epsilon\}$ be the event that X has a jump exceeding ϵ on the interval $[0,t]$. Similarly let $B_\epsilon := \cup_{s\in[0,t]}\{X_s - X_{s-} \geq \epsilon\}$. (If we only have X càdlàg a.s.-P, then these are taken on the a.s. subset Ω_0 of Ω on which this property holds.) On the one hand, by right continuity (outside a P-negligible set) $A_\epsilon \subset \cup_{n\in\mathbb{N}}\cap_{N\in\mathbb{N}}\cup_{\{r,s\}\subset(\mathbb{Q}\cap[0,t])\cup\{t\}, r<s, s-r<1/N}\{X_s - X_r > \epsilon + 1/n\}$. On the other hand, by the càdlàg property, for each $n \in \mathbb{N}$ (outside a P-negligible set) $F_n := \cap_{N\in\mathbb{N}}\cup_{\{r,s\}\subset(\mathbb{Q}\cap[0,t])\cup\{t\}, r<s, s-r<1/N}\{X_s - X_r > \epsilon + 1/n\} \subset B_{\epsilon+1/n}$. Therefore $\cup_{n\in\mathbb{N}}F_n \subset \cup_{n\in\mathbb{N}}B_{\epsilon+1/n} = A_\epsilon$ (P-a.s.). Thus (by completeness) $A_\epsilon \in \mathcal{F}_t$.

Hence, with a positive probability, X will attain a new supremum (on $[0,t]$) by a jump in $\overline{X}$, a contradiction.

Finally, suppose $X(T_x) = x$ a.s.-P on $\{T_x < \infty\}$ for each $x > 0$. Then the supremum process $\overline{X}$ is a.s. continuous. Indeed, suppose not. Then with a positive probability $\overline{X}$ would have a jump, and therefore, for a pair of rationals $0 < r_1 < r_2$, with a positive probability there would be a jump of $\overline{X}$ over (r_1, r_2), whence on this event $X(T_{(r_1+r_2)/2}) \geq r_2 > (r_1 + r_2)/2$, a contradiction. By the equivalence which we have just proven, it follows that a.s.-P X has no positive jumps. □

The first main step towards the proof of Theorem 3 is the following:

Proposition 7 (P-triviality of $X(T_x)$)**.** *$X(T_x)$ on $\{T_x < \infty\}$ is a P-trivial random variable for each $x > 0$ iff either one of the following mutually exclusive conditions obtains:*

(a) X has no positive jumps (P-a.s.) (equivalently: $\lambda((0,\infty)) = 0$).

(b) X is compound Poisson and for some $h > 0$, $\mathrm{supp}(\lambda) \subset \mathbb{Z}_h$ and $\mathrm{supp}(\lambda|_{\mathcal{B}((0,\infty))}) = \{h\}$.

If so, then $X(T_x) = x$ on $\{T_x < \infty\}$ for each $x \geq 0$ (P-a.s.) under (a) and $X(T_x) = h\lceil x/h \rceil$ on $\{T_x < \infty\}$ for each $x \geq 0$ (P-a.s.) under (b).

The main idea of the proof of Proposition 7 is to appeal first to Lemma 6 in order to get (a) and then to treat separately the compound Poisson case; in all other instances the Lévy-Itô decomposition and the well-established path properties of Lévy processes yield the claim. Intuitively, for a Lévy process to cross over every level in a non-random fashion, either it does so necessarily continuously when there are no positive jumps (cf. also [8, p. 274, Proposition 6.1.2]), or, if there are, then it must be forced to live on the lattice $\mathbb{Z}_h$ for some $h > 0$ and only jump up by h. Formally:

Proof. Assume, without loss of generality, that X is càdlàg with certainty (rather than just P-a.s.). Clearly the conditions are mutually exclusive, sufficiency and the final remark obtain by sample path right continuity. With regard to (a) note also [12, p. 346, Remark 46.1]. For necessity observe as follows. Suppose first that for each $x > 0$, $X(T_x) = x$ (P-a.s.) on $\{T_x < \infty\}$. Then by Lemma 6 (a) must hold. There remains the case, when for some $x > 0$, there is a non-random $f(x)$ with $f(x) = X(T_x) > x$ (P-a.s. and with a positive probability) on $\{T_x < \infty\}$. In particular, X must have positive jumps, and for some $a > 0$, $\lambda((a,\infty)) > 0$. Use again the Lévy-Itô decomposition as in Lemma 6 with Z denoting the supremum process of $|X^1 + X^3|$.

(1) Suppose first λ has infinite mass or $\sigma^2 > 0$, or if this fails (with 0 as the cut-off function) $\mu \neq 0$.

By right continuity there is a $t > 0$ with $\mathsf{P}(\{Z_t < a/4\}) > 0$. Now let T be the first jump time of X^2. On the event $C := \{T < t\} \cap \{Z_t < a/4\}$ which transpires with a positive probability, $(X^1 + X^3)(T)$ is not P-trivial. Indeed, note that $T \sim \mathrm{Exp}(\lambda((a,\infty)))$ and suppose *per absurdum* that $(X^1 + X^3)(T)$ were to be P-trivial on the event C, i.e. there would be a (necessarily unique) $b \in (-a/4, a/4)$ such that $(X^1 + X^3)(T) = b$ a.s.-P on C, i.e. $\mathsf{P}(\{(X^1 + X^3)(T) = b\} \cap C) = \mathsf{P}(C)$.

By conditioning on T, and using independence, one obtains (via Proposition 11 — applied to $\sigma(T) \ni Y := T : (\Omega, \mathcal{F}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ (discarding, without loss of generality, the P-negligible set $\{T = \infty\}$) and $\sigma(T) \perp Z := X^1 + X^3 : (\Omega, \mathcal{F}) \to (\mathbb{D}, \mathcal{H})$, where $\mathcal{H}$ is the σ-algebra of all evaluation maps on the space of càdlàg paths $\mathbb{D}$, and with $f : \mathbb{R} \times \mathbb{D} \to \mathbb{R}$ given by (in the obvious notation) $f(s, \omega) := \mathbb{1}_{\{b\}}(\omega(s))\mathbb{1}_{[0,t)}(s)\mathbb{1}_{[0,a/4)}(\overline{\omega}(s))$ – the latter clearly $\mathcal{B}(\mathbb{R}) \otimes \mathcal{H}/\mathcal{B}(\mathbb{R})$-measurable, in particular on account of [7, p. 5, 1.14 Remark] and since $(\omega \mapsto \overline{\omega})$ is $\mathcal{H}/\mathcal{H}$-measurable):

$$\int_0^t ds\beta e^{-\beta s}\mathsf{P}(\{(X^1 + X^3)(s) = b\} \cap \{Z_t < a/4\}) = \int_0^t ds\beta e^{-\beta s}\mathsf{P}(\{Z_t < a/4\}).$$

Hence, Lebesgue-a.e. in $s \in (0, t)$, a.s. on $\{Z_t < a/4\}$, $(X^1 + X^3)(s) = b$. So a.s. on $\{Z_t < a/4\}$, on a dense countable subset of $(0, t)$, $X^1 + X^3 = b$.[2] Thus by right continuity a.s. on $\{Z_t < a/4\}$, $X^1 + X^3 = b$ everywhere on $[0, t)$. And since the set of jumping times of $X^1 + X^3$ is dense, a.s., by [12, p. 136, Theorem 21.3] when λ has infinite mass or since its sample paths have locally infinite variation, a.s., by [12, p. 140, Theorem 21.9(ii)] when $\sigma^2 > 0$, or, finally, since $X^1 + X^3$ has no non-degenerate intervals of constancy, a.s., when $\sigma^2 = 0$, $\lambda(\mathbb{R}) < \infty$ but the drift is non-zero — this yields a contradiction (on our event of positive probability there are *no* jump times on the whole of the interval $[0, t)$, the path has *zero* variation over $[0, t)$ and is, moreover, *constant* thereon). We have thus established that $(X^1 + X^3)(T)$ is *not* P-trivial on the event C. On the other hand the first jump of X^2, $X^2(T)$, is independent of $(X^1 + X^3)(T)$ conditionally on C.[3] The support of their sum $X(T) = (X^1 + X^3)(T) + X^2(T)$ on C, is therefore the closure of the sum of their respective supports [12, p. 148, Lemma 24.1] and as such contains at least two points. It follows that, on the stipulated event of positive probability, which is contained in $\{T_{a/2} < \infty\}$ and on which $T_{a/2} = T$, $X(T_{a/2}) = X(T)$ is not P-trivial, a contradiction.

(2) Suppose now $\sigma^2 = 0$ and λ finite, with the drift being 0.

Suppose *per absurdum* that the support of $\lambda|_{\mathcal{B}((0,\infty))}$ contains at least two points $b < c$, say. Choose $\delta < b/2$ small enough such that $B(b, \delta) \cap B(c, \delta) = \emptyset$. λ must charge both these open balls, and hence the first jump can be in either one, each with a positive probability. Thus $X(T_{b/2})$ would not be P-trivial on the event $\{T_{b/2} < \infty\}$, a contradiction. Plainly, then, the support of $\lambda|_{\mathcal{B}((0,\infty))}$ is $\{h\}$ for some $h > 0$. It remains to show that λ is in fact supported by $\mathbb{Z}_h$. To see this, suppose it were not. Then there would be an $x < 0$ and a $\delta > 0$, with $B(x, \delta)$ having a non-empty intersection with the support of λ and an empty intersection with $\mathbb{Z}_h$. With a positive probability X would jump into $B(x, \delta)$ and then have a sequence of h-jumps upwards going above h for the first time at

[2] We find the dense countable subset by taking for each rational $r \in (0, t)$ an $x_n^r \in B(r, 1/n)$ for which a.s. on $\{Z_t < a/4\}$, $(X^1 + X^3)(x_n^r) = b$ (which we can of course do). Then it is a simple matter of commuting the a.s. property over a countable collection of propositions, which again we can do.

[3] Simply note the equality (for Borel A and B):
$\mathsf{P}(\underbrace{\{T < t\} \cap \{Z_t < a/4\}}_{I} \cap \underbrace{\{X^2(T) \in A\}}_{II} \cap \underbrace{\{(X^1 + X^3)(T) \in B\}}_{I}) \times \mathsf{P}(\underbrace{\{T < t\} \cap \{Z_t < a/4\}}_{II}) = \mathsf{P}(\{T < t\} \cap \{Z_t < a/4\} \cap \{(X^1 + X^3)(T) \in B\}) \times \mathsf{P}(\{T < t\} \cap \{Z_t < a/4\} \cap \{X^2(T) \in A\})$.

a level different from h. With a positive probability, X also goes above h by making its first jump to h, a contradiction.

The proof is complete. $\square$

The second (and last) main step towards the proof of Theorem 3 consists in taking advantage of the temporal and spatial homogeneity of Lévy processes. Thus the condition in Proposition 7 is strengthened to one in which the P-triviality of the position at first passage is required of one only $x > 0$, rather than all. To shorten notation let us introduce:

Definition 8. *For $x \in \mathbb{R}$, let $\mathsf{Q}^x := X(T_x)_\star \mathsf{P}(\cdot \cap \{T_x < \infty\})$ be the (possibly subprobability) law of $X(T_x)$ on $\{T_x < \infty\}$ under P on the space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We also introduce the set*

$$\mathcal{A} := \{x \in \mathbb{R} : \mathsf{Q}^x \text{ is a weighted (possibly by 0) } \delta\text{-distribution}\}.$$

Remark 9. *Clearly $(-\infty, 0] \subset \mathcal{A}$ and for each $a \in \mathcal{A}$, there exists an $f(a)$ such that $\mathsf{Q}^a = \mathsf{P}(T_a < \infty)\delta_{f(a)}$. $f(a)$ is unique, whenever $\mathsf{P}(T_a < \infty) > 0$.*

With this at our disposal, we can formulate our claim as:

Proposition 10. *Suppose $\mathcal{A} \cap \mathbb{R}^+ \neq \emptyset$. Then $\mathcal{A} = \mathbb{R}$.*

The proof will proceed in several steps, but the gist of it consists in establishing the intuitively appealing identity $\mathsf{Q}^b(A) = \int d\mathsf{Q}^c(x_c)\mathsf{Q}^{b-x_c}(A - x_c)$ for Borel sets A and $c \in (0, b)$ (where Q^c must be completed). This is used to show that $\mathcal{A}$ is dense in the reals, and then we can appeal to quasi-left-continuity to conclude the argument. The main argument is thus fairly short, and a substantial amount of time is spent on measurability issues.

Proof. Given $\mathcal{A} \cap \mathbb{R}^+ \neq \emptyset$, we wish to show the inclusion $\mathbb{R}^+ \subset \mathcal{A}$. Assume, again without loss of generality, that X is càdlàg with certainty (rather than just P-a.s.).

(I) First observe that $\mathsf{P}(T_x = \infty) = 1$ for some $x > 0$, precisely when $\mathsf{P}(T_x = \infty) = 1$ for all $x > 0$, by the strong Markov property of Lévy processes. Therefore it is sufficient to consider the case when $\mathsf{P}(T_x < \infty) > 0$ for all $x \in \mathbb{R}$.

(II) Claim: if $b \in \mathcal{A}$, then for every $c \in (0, b)$:
   (a) $c \in \mathcal{A}$ or
   (b) $(0, b - c] \cap \mathcal{A} \neq \emptyset$.

   To show this let $A \in \mathcal{B}(\mathbb{R})$. Then:

$$\begin{aligned}
\mathsf{Q}^b(A) &= \mathsf{E}[\mathbb{1}_A \circ X(T_b)\mathbb{1}_{\{T_b<\infty\}}] \\
&= \mathsf{E}[\mathbb{1}_{\{\overset{\triangle}{X}(\overset{\triangle}{T}_{b-X(T_c)})+X(T_c)\in A\}}\mathbb{1}_{\{\overset{\triangle}{T}_{b-X(T_c)}<\infty\}}\mathbb{1}_{\{T_c<\infty\}}] \\
&= \mathsf{P}(T_c<\infty)\times \mathsf{E}^{\mathsf{P}(\cdot|\{T_c<\infty\})}\left[\mathsf{E}^{\mathsf{P}(\cdot|\{T_c<\infty\})}\left[\mathbb{1}_{\{\overset{\triangle}{X}(\overset{\triangle}{T}_{b-X(T_c)})+X(T_c)\in A\}}\mathbb{1}_{\{\overset{\triangle}{T}_{b-X(T_c)}<\infty\}}|\mathcal{F}'_{T_c}\right]\right] \\
&= \int d\mathsf{Q}^c(x_c)\mathsf{Q}^{b-x_c}(A-x_c) = \mathsf{P}(T_b<\infty)\delta_{f(b)}(A).
\end{aligned}$$

where the first inequality is just the definition of Q^b; in the second the quantities with a $\triangle$ pertain to the process $(X(T_c+t)-X(T_c))_{t\geq 0}$ on $\{T_c<\infty\}$; the third is the consistency property of conditional expectations where $\mathcal{F}'_{T_c}$ is $\mathcal{F}_{T_c}$ lowered onto $\{T_c<\infty\}$; the fourth is the strong Markov property [9, p. 68, Theorem 3.1] using Proposition 12; and the last equality follows from $b\in\mathcal{A}$.

To be sufficiently unapologetic in this argument, we specify precisely how Proposition 12 is applied, this not being completely trivial. The probability space we will be working on is $(\{T_c<\infty\},\mathcal{F}_{\{T_c<\infty\}},\mathsf{P}(\cdot|\{T_c<\infty\}))$ and it is complete, since $(\Omega,\mathcal{F},\mathsf{P})$ is. The random elements are $Z:=(X(T_c+t)-X(T_c))_{t\geq 0}\perp\mathcal{F}'_{T_c}$ living in the canonical space of càdlàg paths $(\mathbb{D},\mathcal{H})$, where $\mathcal{H}$ is the σ-field of all evaluation maps, and $Y:=X(T_c)\in\mathcal{F}'_{T_c}$ living in $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Measurability of $X(T_c)$ is a consequence of [7, p. 9, 2.18 Proposition] and the début theorem [6, p. 101, Theorem 6.7] and measurability of Z follows similarly. The mapping $f:\mathbb{R}\times\mathbb{D}\to\mathbb{R}$ is given by (in the obvious notation) $(x,\omega)\mapsto\mathbb{1}_A(x+\omega(T_{b-x}(\omega)))\mathbb{1}_{[0,\infty)}(T_{b-x}(\omega))$, where we let $\omega(\infty)=\omega(0)$ for definitness.[4] We shall show that f is $(\mathcal{B}(\mathbb{R})\otimes\mathcal{H})^\star/\mathcal{B}(\mathbb{R})$ measurable.

First note that:

(a) $(x,\omega)\mapsto\omega+x$ is $\mathcal{B}(\mathbb{R})\otimes\mathcal{H}/\mathcal{H}$-measurable (in fact continuous, compare [5, p. 328, 1.17 Proposition & p. 329, 1.23 Proposition]), hence $(\mathcal{B}(\mathbb{R})\otimes\mathcal{H})^\star/\mathcal{H}^\star$-measurable, by [11, (2) on p. 23] (i.e. it is universally measurable).

(b) By the début theorem for every $b\in\mathbb{R}$, T_b is a stopping time of the augmented right-continuous modification of the canonical filtration on $\mathbb{D}$ and hence $(\omega\mapsto T_b(\omega))$ is $\mathcal{H}^\star/\mathcal{B}([0,\infty])$-measurable.

It follows that $(x,\omega)\mapsto T_b(\omega+x)=T_{b-x}(\omega)$ is $(\mathcal{B}(\mathbb{R})\otimes\mathcal{H})^\star/\mathcal{B}([0,\infty])$-measurable (as a composition). Next:

(a) $(x,\omega)\mapsto(\omega,\mathbb{1}_{[0,\infty)}(T_{b-x}(\omega))T_{b-x}(\omega))$ is $(\mathcal{B}(\mathbb{R})\otimes\mathcal{H})^\star/\mathcal{H}\otimes\mathcal{B}(\mathbb{R}_+)$-measurable.

(b) $(\omega,t)\mapsto\omega(t)$ is $\mathcal{H}\otimes\mathcal{B}(\mathbb{R}_+)/\mathcal{B}(\mathbb{R})$-measurable (indeed, if X is the coordinate process on $\mathbb{D}$, then this is the mapping $(\omega,t)\mapsto X(\omega,t)$, which is measurable by [7, p. 5, 1.13 Proposition]).

Therefore $(x,\omega)\mapsto\omega(T_{b-x}(\omega))$ is $(\mathcal{B}(\mathbb{R})\otimes\mathcal{H})^\star/\mathcal{B}(\mathbb{R})$-measurable (as a composition, with the above convention for $\omega(\infty)$). The required measurability of f now follows from measurability of addition and multiplication.

We are now in a position to apply Proposition 12. We have: $\mathsf{P}(T_c<\infty)\mathsf{E}^{\mathsf{P}(\cdot|\{T_c<\infty\})}[\mathsf{E}^{\mathsf{P}(\cdot|\{T_c<\infty\})}[f\circ(Y,Z)|\mathcal{F}'_{T_c}]]=\mathsf{P}(T_c<\infty)\mathsf{E}^{\mathsf{P}(\cdot|\{T_c<\infty\})}[(y\mapsto\mathsf{E}^{\mathsf{P}(\cdot|\{T_c<\infty\})}[f\circ(y,Z)])\circ X(T_c)]=\int d\mathsf{Q}^c(y)\mathsf{E}^{\mathsf{P}(\cdot|\{T_c<\infty\})}[f\circ(y,Z)]$. Here we let, as is customary, Q^c also denote the completion of Q^c, which we need, since we only know that the integrand is measurable

[4] This notation is not to be confused with the one from Remark 9 and at any rate is of a provisional nature — introduced solely for the purposes of establishing how Proposition 12 is applied. Moreover, the context will always make it clear which f we are referring to.

with respect to this completion. Now, by the strong Markov property Z is also identical in law under the measure $\mathsf{P}(\cdot|\{T_c < \infty\})$ to X under the measure P on the space $(\mathbb{D}, \mathcal{H})$ and hence on the space $(\mathbb{D}, \mathcal{H}^\star)$ /the extension of a law to the universal completion being unique [11, (1) on p. 23]/. Moreover, for any real d and Borel set D, the mapping $g_{d,D} : \mathbb{D} \to \mathbb{R}$ given by $(\omega \mapsto \mathbb{1}_D(\omega(T_d(\omega)))\mathbb{1}_{[0,\infty)}(T_d(\omega)))$ is $\mathcal{H}^\star/\mathcal{B}(\mathbb{R})$-measurable, by the same reasoning as above. Hence $\mathsf{E}^{\mathsf{P}(\cdot|\{T_c<\infty\})}[f \circ (y, Z)] = \mathsf{E}^{\mathsf{P}(\cdot|\{T_c<\infty\})}[\mathbb{1}_{A-y} \circ \overset{\triangle}{X}(\overset{\triangle}{T}_{b-y})\mathbb{1}_{[0,\infty)} \circ \overset{\triangle}{T}_{b-y}] = \mathsf{E}^{\mathsf{P}(\cdot|\{T_c<\infty\})}[g_{b-y,A-y} \circ Z] = \mathsf{E}^{\mathsf{P}}[g_{b-y,A-y} \circ X] = \mathsf{Q}^{b-y}(A - y)$.

From the equalities $\mathsf{Q}^b(A) = \int d\mathsf{Q}^c(x_c)\mathsf{Q}^{b-x_c}(A - x_c) = \mathsf{P}(T_b < \infty)\delta_{f(b)}(A)$, we see that Q^c-a.e. in $x_c \in \mathbb{R}$, Q^{b-x_c} assigns all its mass to $f(b)-x_c$. (Suppose not, then with Q^c-positive measure in x_c, $\mathsf{Q}^{b-x_c}(\mathbb{R}\backslash\{f(b) - x_c\}) > 0$, and hence $Q^b(\mathbb{R}\backslash\{f(b)\}) > 0$, a contradiction.) Therefore $c \in \mathcal{A}$, or Q^c cannot ascribe all its mass to $f(b)$ and hence $\mathsf{Q}^c([c, b)) > 0$ (notice that (*) Q^c assigns all its mass to $[c, b) \cup \{f(b)\}$). In the latter case, for some $x_c \in [c, b)$, Q^{b-x_c} is a weighted δ-distribution on $f(b) - x_c$. Therefore $b - x_c \in \mathcal{A} \cap (0, b - c]$.

(III) Let $x_0 := \inf \mathcal{A} \cap \mathbb{R}^+$. By the previous item (applied to some $[x_0, \infty) \cap \mathcal{A} \ni b < 3x_0/2$ and $c = 3x_0/4$, say), $x_0 = 0$. Therefore there exists a decreasing sequence x_n in $\mathcal{A} \cap \mathbb{R}^+$ converging to 0.

(IV) Claim: $\mathcal{A}$ is dense in $\mathbb{R}$. If $f(x_n) \to 0$ as $n \to \infty$, this is immediate, since, (**) with any $x \in \mathcal{A}$, $[x + nf(x), (n + 1)f(x)] \subset \mathcal{A}$ ($n \in \mathbb{N}_0$), by the strong Markov property. Suppose the nonincreasing sequence $(n \mapsto f(x_n))$ does not converge to 0. Then there is an $\epsilon > 0$ and a natural N, such that $f(x_n) \geq \epsilon$ and $x_n < \epsilon$ for all $n \geq N$. In particular $f(x_n) = f(x_N)$ for all $n \geq N$ (by (*)). Therefore $[x_n, f(x_N)] \subset \mathcal{A}$ for all $n \geq N$ by (**). Therefore $[0, f(x_N)] \subset \mathcal{A}$ and upon exceeding any positive level less than or equal to $f(x_N)$ we land at $f(x_N)$ a.s. Hence, by the strong Markov property, $\mathcal{A} = \mathbb{R}$.

(V) So we may assume $\mathcal{A}$ is dense. Now we use quasi-left-continuity of Lévy processes [2, p. 21, Proposition 7] as follows. Take any $x \in \mathbb{R}^+$ and a sequence $\mathcal{A} \cap (0, x) \supset x_n \uparrow x$. Introduce the $\mathbb{F}$-stopping time $S := \inf\{t \geq 0 : \overline{X}_t \geq x\}$. We then have $T_{x_n} \uparrow S$ (as $n \to \infty$) a.s.-P. By quasi-left-continuity, it follows that $\lim_{n\to\infty} X(T_{x_n}) = X(S)$ a.s.-P on $\{S < \infty\}$. Therefore $S = T_x$ a.s.-P on $\{S < \infty\}$ (and hence on $\{T_x < \infty\}$), and, moreover $X(T_x) = \lim_{n\to\infty} f(x_n)$ a.s.-P on $\{T_x < \infty\}$. But this means, precisely, that $x \in \mathcal{A}$.

The proof is complete. □

Finally we can combine the above into a proof of Theorem 3.

Proof. (Of Theorem 3.) The statement is essentially contained in Propositions 7 and 10. We only have to worry about (c) and (d), since so far we have only considered the stopping times T_x. Thus, (c) implies for some $f(x) > 0$, $X(\hat{T}_x) = f(x)$ a.s.-P on $\{\hat{T}_x < \infty\}$, therefore $X(T_{f(x)}) = f(x)$ a.s.-P on $\{T_{f(x)} < \infty\}$ and hence (a). Conversely, (e) implies (d) by right continuity. □

4. Conclusion

Theorem 3 characterizes the class of Lévy processes for which overshoots are known *a priori* and are non-random. Moreover, the original motivation for this investigation is validated by the fact that upwards-skip-free Lévy chains admit for a fluctuation theory just as explicit and almost (but not entirely) analogous to the spectrally negative case — the publication of these findings, however, is deferred to another occasion.

Appendix: Lemma on conditioning

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space.

Proposition 11 (Lemma on conditioning)**.** *Let $Y : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ and $Z : (\Omega, \mathcal{F}) \to (T, \mathcal{T})$ be two random elements, and $\mathcal{G}$ any sub-σ-algebra of $\mathcal{F}$, such that $\sigma(Y) \subset \mathcal{G}$ and $\sigma(Z) \perp \mathcal{G}$. Let f be any bounded (or nonnegative, or nonpositive) $\mathcal{S} \otimes \mathcal{T}/\mathcal{B}(\mathbb{R})$-measurable mapping. Then for any $y \in S$, $f \circ (y, Z)$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable, $(y \mapsto \mathsf{E}[f \circ (y, Z)])$ is $\mathcal{S}/\mathcal{B}(\mathbb{R})$-measurable and, a.s.-$\mathsf{P}$,*

$$\mathsf{E}[f \circ (Y, Z)|\mathcal{G}] = (y \mapsto \mathsf{E}[f \circ (y, Z)]) \circ Y.$$

Proof. In its entirety by a usual π/λ-argument and approximation techniques. □

There is an extension of this proposition, which allows for completions, to wit:

Proposition 12 (Lemma on conditioning, bis)**.** *Assume now $(\mathcal{F}, \mathsf{P})$ is complete. Let $Y : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ and $Z : (\Omega, \mathcal{F}) \to (T, \mathcal{T})$ again be two random elements, and $\mathcal{G}$ any sub-σ-algebra of $\mathcal{F}$, such that $\sigma(Y) \subset \mathcal{G}$ and $\sigma(Z) \perp \mathcal{G}$. Let f be any bounded (or nonnegative, or nonpositive) $\overline{\mathcal{S} \otimes \mathcal{T}}^{(Y,Z)_\star \mathsf{P}}/\mathcal{B}(\mathbb{R})$-measurable mapping. Then:*

(i) (Y, Z) is $\mathcal{F}/\overline{\mathcal{S} \otimes \mathcal{T}}^{(Y,Z)_\star \mathsf{P}}$-measurable,
(ii) Y (resp. Z) is $\mathcal{F}/\overline{\mathcal{S}}^{Y_\star \mathsf{P}}$-measurable (resp. $\mathcal{F}/\overline{\mathcal{S}}^{Z_\star \mathsf{P}}$-measurable),
(iii) $(\overline{\mathcal{S}}^{Y_\star \mathsf{P}}, Y_\star \mathsf{P})$-a.s. in $y \in S$, $f \circ (y, Z)$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable,
(iv) $(y \mapsto \mathsf{E}[f \circ (y, Z)])$ is $\overline{\mathcal{S}}^{Y_\star \mathsf{P}}/\mathcal{B}(\mathbb{R})$-measurable

and, a.s.-P,

$$\mathsf{E}[f \circ (Y, Z)|\mathcal{G}] = (y \mapsto \mathsf{E}[f \circ (y, Z)]) \circ Y. \tag{1}$$

Proof. This is a bit more delicate and so we give the essential steps. Throughout we use the image-measure theorem [4, p. 121, Theorem 4.1.11].

First note that (Y, Z) is $\mathcal{F}/\mathcal{S} \otimes \mathcal{T}$-measurable, hence $\mathcal{F}/\overline{\mathcal{S} \otimes \mathcal{T}}^{(Y,Z)_\star \mathsf{P}}$-measurable, since $\mathcal{F}$ is P-complete. Similarly for Y and Z. (In both cases apply a generating class argument, see also [6, p. 21, Exercise 8]). Thus we have (i) and (ii).

Next, the measure spaces $(S, \overline{\mathcal{S}}^{Y_\star \mathsf{P}}, Y_\star \mathsf{P})$ and $(T, \overline{\mathcal{T}}^{Y_\star \mathsf{P}}, Z_\star \mathsf{P})$ are complete and, by [13, p. 543, Theorem 23.23], $\overline{\overline{\mathcal{S}}^{Y_\star \mathsf{P}} \otimes \overline{\mathcal{T}}^{Z_\star \mathsf{P}}}^{Y_\star \mathsf{P} \times Z_\star \mathsf{P}} = \overline{\mathcal{S} \otimes \mathcal{T}}^{(Y,Z)_\star \mathsf{P}}$, since $Y_\star \mathsf{P} \times Z_\star \mathsf{P} = (Y, Z)_\star \mathsf{P}$, by independence

of Y and Z. Thus (iii) (resp. (iv)) follows from [13, p. 545, Theorem 23.25(b)] and (ii) (resp. by Tonelli's theorem [13, p. 546, Theorem 23.26]).

Finally we wish to establish (1). Linearity of expectation, monotonicity and the usual approximation techniques make sure that we need only consider $f = \mathbb{1}_\Lambda$ with $\Lambda \in \overline{\mathcal{S} \otimes \mathcal{T}}^{(Y,Z)_\star \mathsf{P}}$. An additional π/λ-argument, again coupled with linearity of expectation and monotonicity, shows that it is enough to consider Λ belonging to the π-system $\{A \times B : (A, B) \in \mathcal{S} \times \mathcal{T}\} \cup \mathcal{N}$, where $\mathcal{N}$ is the set of all $(Y, Z)_\star\mathsf{P}$-null sets (i.e. those having measure 0 for the completion of the law $(Y, Z)_\star\mathsf{P}$), and which generates $\overline{\mathcal{S} \otimes \mathcal{T}}^{(Y,Z)_\star \mathsf{P}}$. For Λ belonging to $\{A \times B : (A, B) \in \mathcal{S} \times \mathcal{T}\}$, (1) follows at once by independence and the "taking out what is known" property of conditional expectation. Finally suppose Λ is $(Y, Z)_\star\mathsf{P}$-null. Then the LHS of (1) is equal to 0, a.s.-P. The RHS is nonnegative, a.s.-P. Now, compute its expectation: $\int (Y_\star\mathsf{P})(dy) \int (Z_\star\mathsf{P})(dz) f(y, z) = \int d((Y, Z)_\star\mathsf{P})f = 0$ (where we emphasize that we must actually work with the completions of these laws and for the $Z_\star\mathsf{P}$-integral (resp. $(Y, Z)_\star\mathsf{P}$-integral) we use again [13, p. 545, Theorem 23.25(b)] (resp. [13, p. 543, Theorem 23.23])). Thus also the RHS equals 0, a.s.-P. $\square$

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